# Mathematical Properties of Graphons

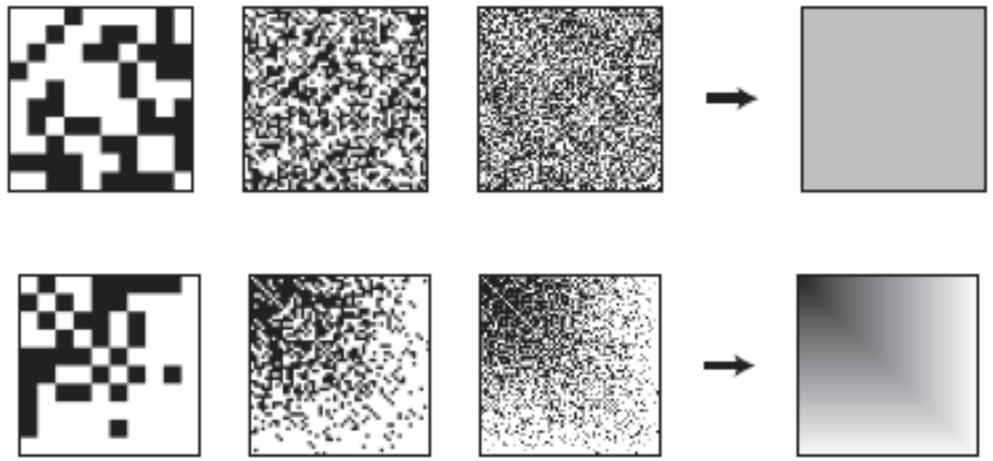
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NeurIPS Education Track 2026

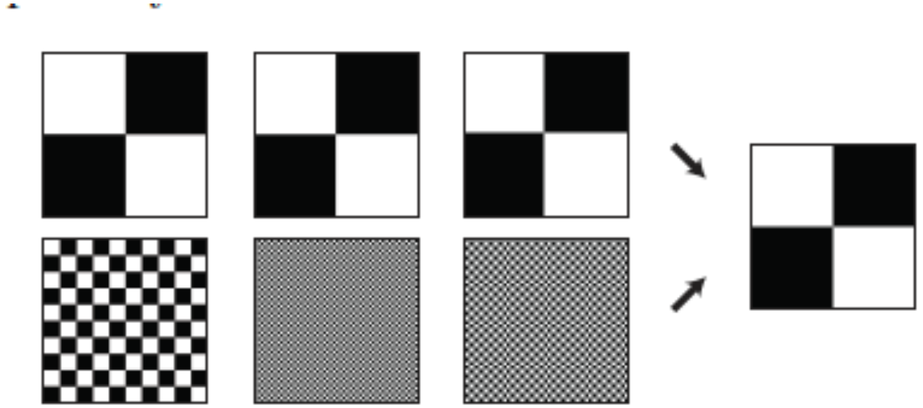
# Graphons are graph limits

- Map the adjacency matrix to the unit square (empirical graphons) and take a limit

Erdos-Renyi graphs



Growing uniform attachment graphs



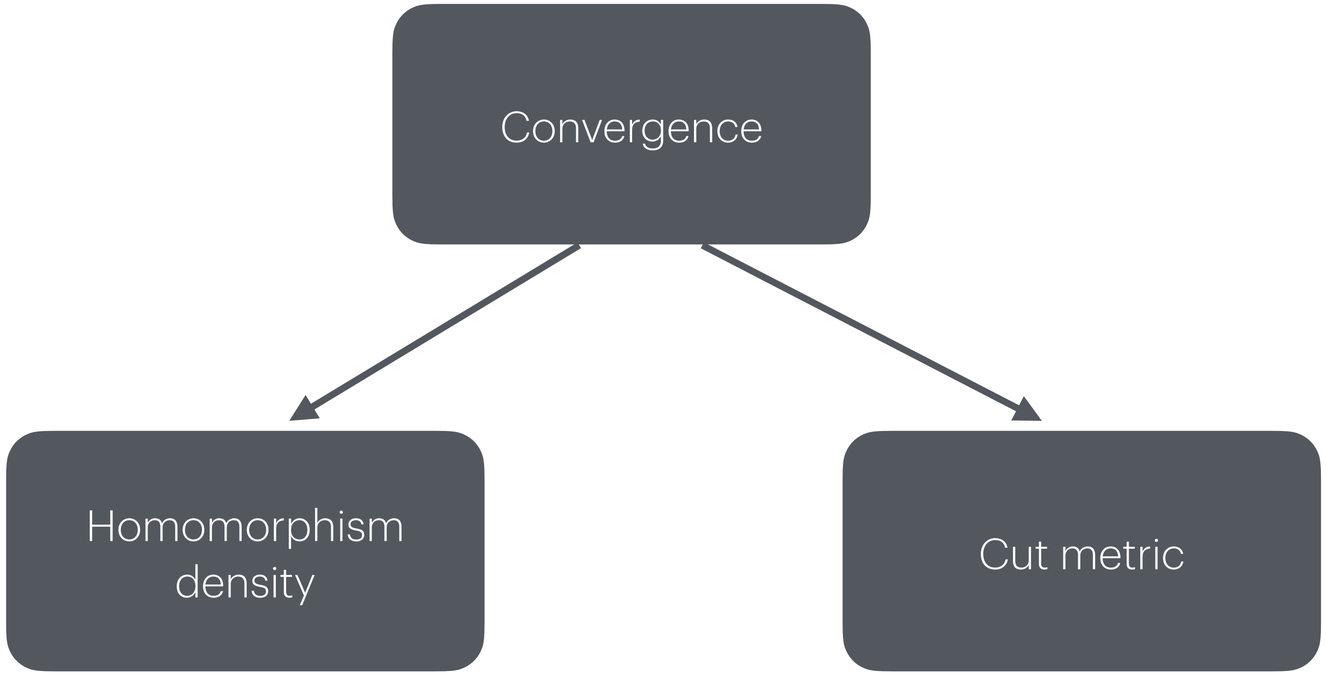
Bi-partite graphs



Glasscock, D. (2015). What is... a graphon. *Notices of the AMS*, *62*(1), 46-48.

# How is convergence defined?

- Sequence of graphs $(G_1, G_2, \ldots, G_n, \ldots)$






Borgs, C., Chayes, J., Lovász, L., Sós, V., & Vesztergombi, K. (2011). Limits of randomly grown graph sequences. *European Journal of Combinatorics*, *32*(7), 985-999

# Homomorphism density



# Graph homomorphism convergence

- Subgraph density convergence

- A graph homomorphism from $F$ to $G$ is a map $f : V(F) \to V(G)$ such that if $uv \in E(F)$ then $f(u)f(v) \in E(G)$ (Maps edges to edges). Let $Hom(F, G)$ be the set of all such homomorphisms and let $\hom(F, G) = |Hom(F, G)|$. Then **homomorphism density** is defined as

- $$t(F, G) = \frac{\hom(F, G)}{|V(G)|^{|V(F)|}}, \quad \text{note that} \quad t(F, G) \in [0,1]$$

- A graph sequence $\{G_n\}_n$ is said to be convergent if $t(F, G_n)$ converges as $n$ goes to infinity for any simple graph $F$.

# Graph homomorphisms

- $t(\bullet\!\!-\!\!\bullet, G) = 2\dfrac{|E(G)|}{n^2}$, where $|E(G)|$ is the number of edges and $n$ is the number of vertices
- This is similar to edge density $2\dfrac{|E(G)|}{n(n-1)}$
- Consider the graph sequence $(G_1, G_2, \ldots)$
- Does the sequence $\left(2\dfrac{|E(G_1)|}{n_1^2}, 2\dfrac{|E(G_2)|}{n_2^2}, \ldots\right)$ converge?
- Does $(t(F, G_1), t(F, G_2), \ldots)$ converge for all simple graphs $F$?

# Graph homomorphisms

- If $(t(F, G_1), t(F, G_2), \ldots)$ converges for all simple graphs $F$, then the graphs converge to a graphon $W$
- $\{G_n\}_n$ converges iff there is a $W$ such that $t(F, G_n) \to t(F, W)$ for all simple graphs $F$.
- $$t(F, W) = \int_{[0,1]^{|V(F)|}} \prod_{ij \in E(F)} W(x_i, x_j)\, dx$$
- And for every $W$ there is a convergent sequence of graphs like above.

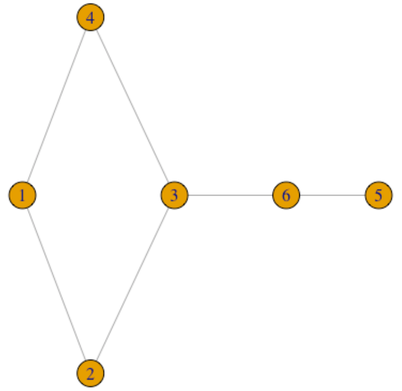
Borgs, C., Chayes, J. T., Lovász, L., Sós, V. T., & Vesztergombi, K. (2008). Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. Advances in Mathematics, 219(6), 1801–1851. https://doi.org/10.1016/j.aim.2008.07.008

# The catch: convergence must hold for every simple graph $F$

# Cut metric



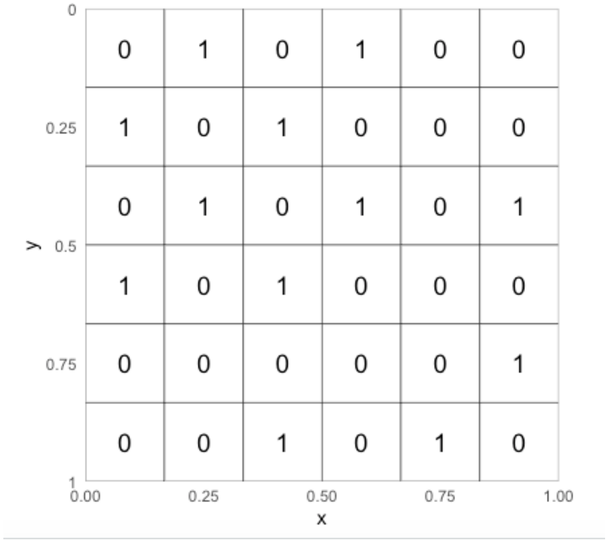
# Three related objects



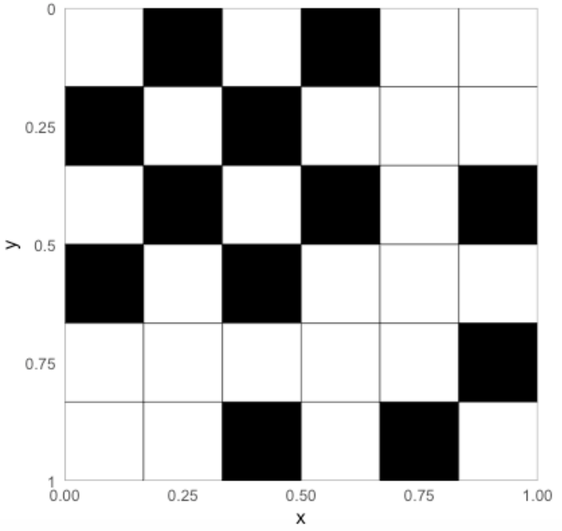

- Adjacency matrix - Empirical graphon - pixel pictures

| | [,1] | [,2] | [,3] | [,4] | [,5] | [,6] |
|---|---|---|---|---|---|---|
| [1,] | 0 | 1 | 0 | 1 | 0 | 0 |
| [2,] | 1 | 0 | 1 | 0 | 0 | 0 |
| [3,] | 0 | 1 | 0 | 1 | 0 | 1 |
| [4,] | 1 | 0 | 1 | 0 | 0 | 0 |
| [5,] | 0 | 0 | 0 | 0 | 0 | 1 |
| [6,] | 0 | 0 | 1 | 0 | 1 | 0 |

Adjacency matrix




Empirical graphon



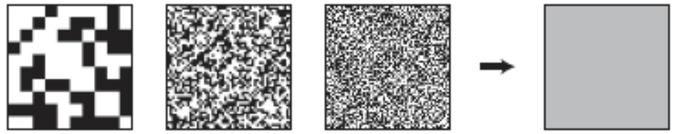

Pixel picture

Matrix gets converted to function on a unit square

1 => black
0 => white

# Convergence - cut metric

- Pixel picture/empirical graphon convergence



- The cut norm of a graphon $W$ is defined as $\|W\|_{\square} = \sup_{S,T} \left| \int_{S \times T} W(x,y)\, dxdy \right|$ where supremum is taken over all measurable sets $S$ and $T$ of $[0,1]$

- Given two graphons $W_1$ and $W_2$, the **cut metric** is defined as $\delta_{\square}(W_1, W_2) = \inf_{\varphi} \left\| W_1 - W_2^{\varphi} \right\|_{\square}$ where $\varphi$ is a measure preserving bijection from $[0,1]$ to $[0,1]$.

- Why $\varphi$? Because nodes can be re-named or permuted.

# Convergence - cut metric

- A sequence of graphs $(G_1, G_2, \ldots, )$ converge to $W$ if

$$\lim_{n \to \infty} \delta_{\square}(W_{G_n}, W) = 0$$

where $W_{G_n}$ is the empirical graphon of graph $G_n$

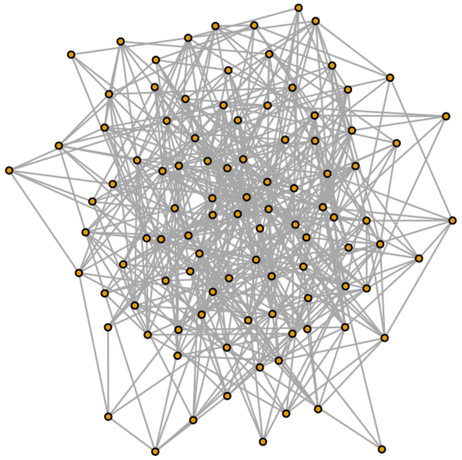
- Convergence in homomorphism density is equivalent to convergence in cut-metric.

Borgs, C., Chayes, J. T., Lovász, L., Sós, V. T., & Vesztergombi, K. (2008). Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. *Advances in Mathematics*, *219*(6), 1801-1851.

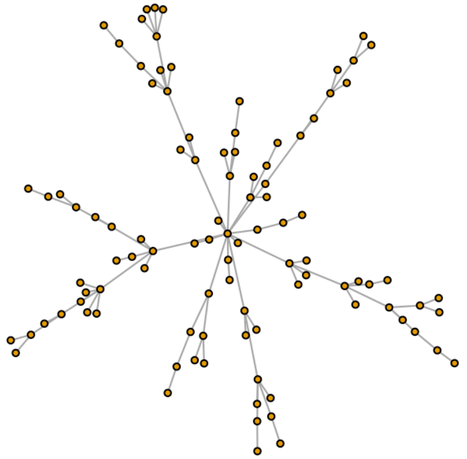
# Dense and sparse graphs

- Consider a graph sequence $\{G_n\}_{n'}$ where $G_n$ has $n$ nodes and $m$ edges.
- Dense graphs: $\liminf_{n\to\infty} \frac{m}{n^2} = c > 0$
  - **Edges grow quadratically with nodes**
- Sparse graphs: $\lim_{n\to\infty} \frac{m}{n^2} = 0$
  - **Edges grow subquadratically with nodes**

Dense



Sparse



# Exchangeability



# A bit on exchangeability

Exchangeability: order of the variables does not change the joint distribution.

| | **Discrete** | **Continuous** |
|---|---|---|
| **1D (Sequences/ Point processes)** | de Finetti (1931) | Bühlmann (1960) |
| **2D (Arrays Joint/ separate)** | Aldous (1981) and Hoover (1979) | Kallenberg (1990) |


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Bühlmann, H. (1960). *Austauschbare stochastische Variabeln und ihre Grenzwertsätze* (Doctoral dissertation, ETH Zurich)

Aldous, D. J. (1981). Representations for partially exchangeable arrays of random variables. *Journal of Multivariate Analysis*, *11*(4), 581-598.

Hoover, D. N. (1979). Relations on Probability Spaces and Arrays of. *Preprint, Institute for Advanced Study,* Princeton

Kallenberg, O. (1990). Exchangeable random measures in the plane. *Journal of Theoretical Probability*, *3*(1), 81-136.

# Exchangeable binary matrices

- Aldous-Hoover representation theorem

$$A_{ij} \mid U_i, U_j, W \sim \text{Ber}(W(U_i, U_j))$$

with $U_i \overset{\text{iid}}{\sim} \text{Unif}(0,1)$, and $W : [0,1]^2 \to [0,1]$ a random function

- Infinite exchangeable binary matrices imply a mixture model representation exist
- For undirected graphs, this mixture is specified by the **graphon** (Borgs et al 2008, Lovász et al 2006)
- For graphs the connection to Aldous-Hoover made by Diaconis & Janson


Borgs, C., Chayes, J. T., Lovász, L., Sós, V. T., & Vesztergombi, K. (2008). Convergent sequences of dense graphs I: Subgraph frequencies, metric properties and testing. *Advances in Mathematics*, *219*(6), 1801-1851.

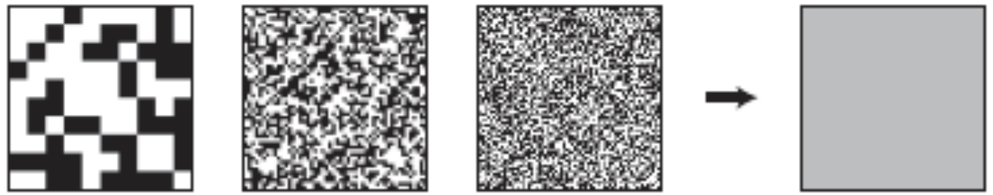
Janson, S., & Diaconis, P. (2008). Graph limits and exchangeable random graphs. *Rendiconti di Matematica e delle sue Applicazioni. Serie VII*, 33-61.

Lovász, L., & Szegedy, B. (2006). Limits of dense graph sequences. *Journal of Combinatorial Theory, Series B*, *96*(6), 933-957.

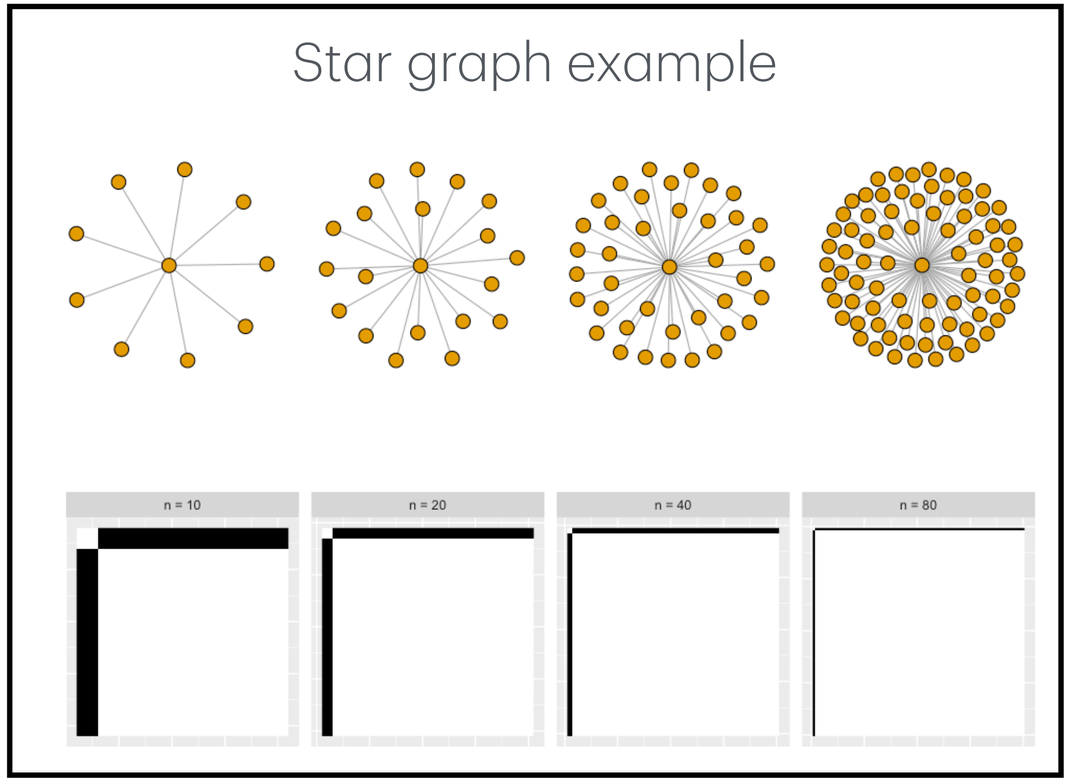
Lovász, L. (2012). *Large networks and graph limits* (Vol. 60). American Mathematical Soc..

# Limitation of traditional graphons

- ***Graphs represented by an exchangeable matrix are either trivially empty or dense***
- These graphons can model dense graphs but not sparse graphs



- Why?
  - The volume from $(x, y, W(x, y))$ converges to $\lim_{n\to\infty} \frac{2m}{n^2}$
  - For sparse graphs $\lim_{n\to\infty} \frac{m}{n^2} = 0 \iff W = 0$



# What happens when $W = 0$?



$W = 0$

- Sample a graph from $W = 0$
- You have a graph with $n$ nodes, but no edges — not representative of sparse graphs

***This is the fundamental problem of graphons!***